§8. Chern-Simons theory and connection on surfaces

To define Chern-Simons theory, we need 3 ingredients:

- compact oriented 3-manifold M
- compact simple gauge group G
- P principal G bundle over M

In these lectures: $G = SU(2)$

$\longrightarrow$ P is topologically trivial ($SU(2)$ simply connected)

Denote by $\mathcal{A}_M$ the space of connections on P.
identify $\mathcal{A}_M$ with $\Omega^1(M, \mathfrak{g})$ Lie-algebra valued 1-forms
coordinate notation: $A_i^a \in \mathcal{A}_M$

tangent to M   Lie-algebra generator

<u>Definition</u> (gauge transformation):

$$\mathcal{G} := \mathrm{Map}(M, G)$$ space of smooth maps from M to G

$$g^* A = g^{-1} A g + g^{-1} dg, \quad A \in \mathcal{A}_M, \; g \in \mathcal{G}$$

infinitesimal g. trf.: $A_i \mapsto A_i - D_i \varepsilon$,
with $D_i \varepsilon = \partial_i \varepsilon + [A_i, \varepsilon]$

Definition (curvature):

$$F_A = dA + A \wedge A \in \Omega^2(M, \mathfrak{g})$$

Definition (Chern-Simons functional):

For $A \in \mathcal{A}_M$ we put

$$CS(A) = \frac{1}{8\pi^2} \int_M \mathrm{Tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right)$$

Proposition 1:

A critical point of the Chern-Simons functional is a flat connection.

Proof:

Consider a one-parameter family of connections $A_t = A + ta$. Then

$$CS(A + ta) = CS(A) + \frac{t}{4\pi^2} \int_M \mathrm{Tr}\left(F_A \wedge a\right) + O(t^2)$$

(exercise)

$\Rightarrow$ CS is critical at $A \Leftrightarrow F_A = 0$.

$\square$

Proposition 2:

Let M be a compact oriented 3-manifold with $\partial M \neq 0$. Then we have

$$CS(g^*A) = CS(A) + \frac{1}{8\pi^2}\int_{\partial M} \mathrm{Tr}\left(A \wedge dg\, g^{-1}\right) - \int_M g^*\sigma$$

where σ is the volume form of $SU(2)$:

$$g^*\sigma = \frac{1}{24\pi^2}\mathrm{Tr}\left(g^{-1}dg \wedge g^{-1}dg \wedge g^{-1}dg\right)$$

Proof:

$$g^*A = g^{-1}Ag + g^{-1}dg$$

$$g^*F_A = g^{-1}F_A g \,, \quad \text{where } F_A = dA + A \wedge A$$

$$\Rightarrow CS(g^*A) = \frac{1}{8\pi^2}\int_M \mathrm{Tr}\left(g^*A \wedge g^*F_A - \frac{1}{3} g^*A \wedge g^*A \wedge g^*A\right)$$

$$= \frac{1}{8\pi^2}\int_M \mathrm{Tr}\Big(A \wedge F_A + \underbrace{g^{-1}dg \wedge g^{-1}F_A g}_{= -dA \wedge dg\, g^{-1} + \cancel{A \wedge A \wedge dg\, g^{-1}}}\Big)$$

$$- \frac{1}{24\pi^2}\int_M \Big(g^*\sigma + g^{-1}dg \wedge \cancel{g^{-1}Ag} \wedge g^{-1}Ag + \cancel{\text{perm}}.$$

$$+ g^{-1}dg \wedge \cancel{g^{-1}dg} \wedge g^{-1}Ag + \text{perm}.\Big)$$

$$= d\left(A \wedge dg g^{-1}\right) - A \wedge dg \cancel{\wedge g^{-1}dg}\, g^{-1}$$

$$- \frac{1}{24\pi^2}\int_M \mathrm{Tr}\left(A \wedge A \wedge A\right)$$

□

Set now $\partial M = \Sigma$ (Riemann surface)

Denote by Q a principal G bundle over Σ.

For $G = SU(2) \longrightarrow Q \cong \Sigma \times SU(2)$, since $SU(2)$ simply connected

Denote by $\mathcal{A}_\Sigma$ the space of connections on Q.

We have $\mathcal{A}_\Sigma \cong \Omega^1(\Sigma, \mathfrak{g})$

On $\mathcal{A}_\Sigma$ there is non-degenerate anti-symmetric bilinear form ω defined by

$$\omega(\alpha, \beta) = -\frac{1}{8\pi^2} \int_\Sigma \operatorname{Tr}(\alpha \wedge \beta), \quad \alpha, \beta \in \Omega^1(\Sigma, \mathfrak{g})$$

with $d\omega = 0$.

$\longrightarrow \mathcal{A}_\Sigma$ has structure of infinite dimensional symplectic manifold.

By Prop. 3 §1 $\exists$ line bundle $\mathcal{L}$ over $\mathcal{A}_\Sigma$ and a connection ∇ on $\mathcal{L}$ s.t. $\omega = c_1(\nabla)$ (first Chern class) $\longrightarrow$ Quantization of $\mathcal{A}_\Sigma$

In the following we shall construct $\mathcal{L}$.

Denote by $\mathcal{G}_\Sigma$ the gauge group of Q.

$\longrightarrow \mathcal{G}_\Sigma \cong \operatorname{Map}(\Sigma, G)$

For $a \in \mathcal{A}_\Sigma$ and $g \in \mathcal{G}_\Sigma$ let A be an extension of a on M and $\tilde{g} : M \longrightarrow G$ an extension of g as a smooth map from M to G.

Set

$$c(a, g) = \exp\left(2\pi\sqrt{-1}\left(CS(\tilde{g}^*A) - CS(A)\right)\right)$$

More explicitly,

$$c(a, g) = \exp 2\pi\sqrt{-1}\left(\int_\Sigma \frac{1}{8\pi^2} \mathrm{Tr}\left(g^{-1}ag \wedge g^{-1}dg\right) - \underbrace{\int_M g^*\sigma}_{\text{Wess-Zumino term}}\right)$$

Summarizing, we have

<u>Proposition 3:</u>

Let M be a compact oriented 3-manifold with boundary Σ. For a connection A of a principal G bundle P over M and a gauge transformation $g \in \mathrm{Map}(M, G)$ we have

$$\exp\left(2\pi\sqrt{-1}\, CS(g^*A)\right) = c(a, g|_\Sigma) \exp\left(2\pi\sqrt{-1}\, CS(A)\right)$$

where $g|_\Sigma$ denotes the restriction of g on Σ.

Let $a \in \mathcal{A}_\Sigma$. Define $L_{\Sigma,a}$ as the set of maps

$f : \mathrm{Map}(\Sigma, G) \longrightarrow \mathbb{C}$ satisfying

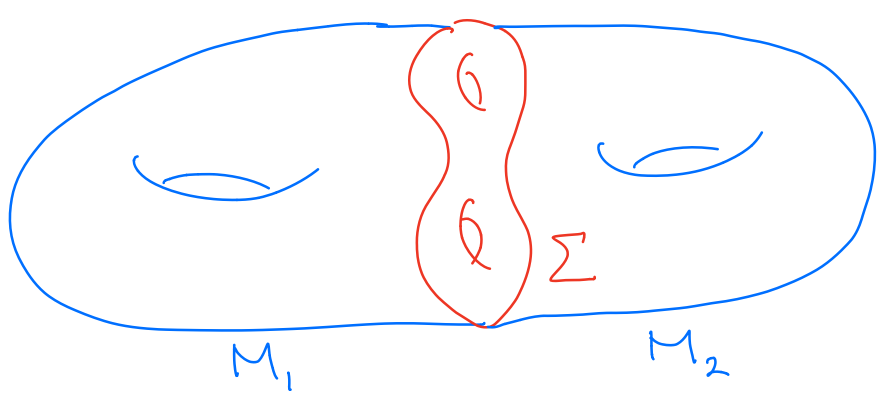
$$f(e \cdot g) = c(a, g) f(e), \quad g \in \mathrm{Map}(\Sigma, G)$$

$\longrightarrow L_{\Sigma, a}$ is 1-dimensional complex vector space with Hermitian inner product.

Prop. 3 $\longrightarrow \exp\left(2\pi\sqrt{-1}\, CS(A)\right) \in L_{\Sigma, a}$

For $-\Sigma$ (Σ with reversed orientation) we have $\quad L_{-\Sigma, a} \cong \overline{L}_{\Sigma, a}$

Let $M = M_1 \cup M_2$ with $\partial M_1 = \Sigma$ and $\partial M_2 = -\Sigma$



Let A be a connection on M and A_1 and A_2 its restrictions on M_1 and M_2.
$\alpha =$ restriction of A on Σ.

$\longrightarrow \exp\left(2\pi\sqrt{-1}\, CS_{M_1}(A_1)\right) \in L_{\Sigma, \alpha}, \; \exp\left(2\pi\sqrt{-1}\, CS_{M_2}(A_2)\right) \in \overline{L}_{\Sigma, \alpha}$

Using the canonical pairing

$$L_{\Sigma,\alpha} \times L_{-\Sigma,\alpha} \longrightarrow \mathbb{C}$$

we get

$$\exp\left(2\pi\sqrt{-1}\, CS_M(A)\right)$$
$$= \left\langle \exp\left(2\pi\sqrt{-1}\, CS_{M_1}(A_1)\right), \exp\left(2\pi\sqrt{-1}\, CS_{M_2}(A_2)\right)\right\rangle$$

Denote by η_t, $0 \leq t \leq 1$ a one-parameter family of connections of a G-bundle Q over Σ.

$\longrightarrow$ regard η as connection over $\Sigma \times [0,1]$

$\longrightarrow$ $CS_{\Sigma\times[0,1]}$ defines a map

$$\exp\left(2\pi\sqrt{-1}\, CS_{\Sigma\times[0,1]}\right) : L_{\eta_0} \rightarrow L_{\eta_1} \qquad (*)$$

Let L_Σ be a topologically trivial line bundle over $\mathcal{A}_\Sigma$. For a path η_t, $0 \leq t \leq 1$, in $\mathcal{A}_\Sigma$

$\xrightarrow{(*)}$ lift to the total space of L_Σ

$\longrightarrow$ connection ∇ on L_Σ with hor. sections given by above lift

$\longrightarrow$ can verify: $c_1(\nabla) = \omega$

Lift action of gauge group $\mathcal{G}_\Sigma$ to L_Σ.

Define
$$\mathcal{M}_\Sigma = \mathcal{A}_\Sigma /\!/ \mathcal{G}_\Sigma = \mu^{-1}(0)/\mathcal{G}_\Sigma$$

(Marsden- Weinstein quotient)

moduli space of flat G- connections on Σ

$\longrightarrow$ complex line bundle $\mathcal{L}$ on $\mathcal{M}_\Sigma$.

The Chern- Simons partition function for a 3 - manifold M is formally written as
$$Z_k(M) = \int_{\mathcal{A}_M/\mathcal{G}} \exp\left(2\pi\sqrt{-1}\, k\, CS(A)\right) \mathcal{D}A \quad (**)$$

Suppose that M is oriented 3- manifold with boundary Σ. Have shown
$$\exp\left(2\pi\sqrt{-1}\, k\, CS(A)\right) \in L_{\Sigma,\alpha}$$

$\mathcal{A}_{M,\alpha}$:= space of G- connections on M whose restriction on $\Sigma = \alpha$

Restrict the path integral in $(**)$ to $\mathcal{A}_{M,\alpha}$

Since $\exp\left(2\pi\sqrt{-1}\, k\, CS(g^*A)\right) = c(g,\alpha)^k \exp\left(2\pi\sqrt{-1}\, k\, CS(A)\right)$

$\rightarrow Z_k(M)$ is section of complex line bundle $\mathcal{L}^{\otimes k}$.